**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

Febuary 2017

**Problem:**

Find the greatest real number $M$ such that

$$(x^2+y^2)^3 \geq M(x^3+y^3)(xy-x-y)$$

for all real numbers $x, y$ satisfying $x+y \geq 0$.

**Solution:** The answer: The greatest $M = 32$.

$x = y = 4$ yields $M \leq 32$. Let us prove that

$$(x^2+y^2)^3 \geq 32(x^3+y^3)(xy-x-y)$$

Let $s = x^2+y^2$ and $t = x+y$. We should show that for all $2s \geq t^2$ and $t \geq 0$ the inequality

$$s^3 \geq 8t(3s-t^2)(t^2-2t-s)$$

holds. Now let $s = rt$. This transforms the required inequality to the inequality

$$r^3 \geq 8(3r-t)(t-2-r)$$

for $2r \geq t \geq 0$.

Since for all $2r \geq t \geq 0$ we have

$$r^3-8(3r-t)(t-2-r) = 8(t-(2r+1))^2+r^3-8r^2+16r-8 \geq r^3-8r^2+16r = r(r-4)^2 \geq 0$$

the proof is completed.