Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2024

Problem:

Find all pairs of positive integers (a, b) such that

$$\frac{10^{a!} - 3^b + 1}{2^a}$$

is a perfect square.

Solution: Answer: $(a, b) = (1, 1)$ and $(1, 2)$.

Assume that

$$\frac{10^{a!} - 3^b + 1}{2^a} = m^2$$

holds for some positive integer m. Let us rewrite the equation as

$$3^b = 10^{a!} - m^2 2^a + 1.$$

If $a = 1$ then clearly the only solutions are $(a, b) = (1, 1)$ and $(1, 2)$. If $a \geq 2$ then $a!$ is even. Then, since 3 divides $10^{a!} - m^2 2^a + 1$, we get $m^2 2^a \equiv 2 \pmod 3$ which means that a is odd. If $a = 3$ then by using modulo 4 we get that b is even. Since $101 \mid 10^6 + 1 = 3^b + m^2 2^a = c^2 + 2d^2$, we have no integer solutions because -2 is not a quadratic residue modulo 101 and $101 \nmid 3^{b/2}$. Therefore, $a \geq 5$. In that case, by using modulo 16 we get that $4 \mid b$ and hence $5 \mid 3^b - 1$. Then $5 \mid m$ and hence $25 \mid 3^b - 1$. Let $b = 4k$, then $25 \mid 81^k - 1$ and from the LTE lemma, $v_5(81^k - 1) = v_5(80) + v_5(k) \geq 2$ and hence $5 \mid k$, we get $5 \mid b$. Hence $3^b \equiv 1 \pmod{11}$ and $11 \mid 10^{a!} - m^2 2^a = x^2 - 2y^2$, but 2 is a not a quadratic residue modulo 11. Thus $11 \mid 10^{a!/2}$, which is a contradiction. Thus, there is no solution for $a \geq 2$.