Bilkent University
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Problem Of The Month

November 2024

Problem:

At the beginning the board contains 77 vectors

$$(1,0,0,\dots,0),\ (0,1,0,\dots,0),\ \dots,\ (0,0,0,\dots,1)$$

each having 77 components. At each step we choose two vectors $(a_1, a_2, \dots, a_{77})$ and $(b_1, b_2, \dots, b_{77})$ written on the board and write their sum $(a_1+b_1, a_2+b_2, \dots, a_{77}+b_{77})$ to the board. Find the minimal number of steps which should be made to get all the vectors

$$(0,1,1,\dots,1),\ (1,0,1,\dots,1),\ \dots,\ (1,1,1,\dots,0).$$

on the board.

Solution: Answer: $3 \cdot 77 - 6 = 225$.

Let us consider more general case when 31 is replaced by $n \geq 3$: At the beginning the board contains n vectors

$$(1,0,0,\dots,0),\ (0,1,0,\dots,0),\ \dots,\ (0,0,0,\dots,1)$$

each having n components and we are going get all the n component vectors

$$(0,1,1,\dots,1),\ (1,0,1,\dots,1),\ \dots,\ (1,1,1,\dots,0).$$

We will show that the minimal number of steps is $3n-6$.

Let us show that $3n-6$ steps are sufficient. Let v_i^n be a vector whose $i-$th coordinate is 1 and all remaining $n-1$ coordinates are 0 and let u_i^n be a vector whose ith coordinate is 0 and all remaining $n-1$ coordinates are 1.

We use Induction over n. If $n=3$ then by applying $v_1^3+v_2^3$, $v_1^3+v_3^3$ and $v_2^3+v_3^3$ we get the u_1^3, u_2^3 and u_3^3 in 3 steps. Assume that for $n=k$ the required vectors can be obtained in

$3k-6$ steps and let $n=k+1$. At the first step by adding v_k^{k+1} and v_{k+1}^{k+1} we get the vector $v_k^{k+1}+v_{k+1}^{k+1} \equiv w_{k,k+1}^{k+1}$. By inductive hypothesis, starting with vectors $v_1^k, v_2^k, \ldots, v_{k-1}^k$ and v_k^k after $3k-6$ steps we can get the vectors $u_1^k, u_2^k, \ldots, u_{k-1}^k$ and u_k^k. If we replace v_1^k by v_1^{k+1}, v_2^k by $v_2^{k+1}, \ldots, v_{k-1}^k$ by v_{k-1}^{k+1} and v_k^k by $w_{k,k+1}^{k+1}$and apply the same $3k-6$ steps then we will get the vectors $u_1^{k+1}, u_2^{k+1}, \ldots, u_{k-1}^{k+1}$ and the vector $\bar{w}_{k,k+1}^{k+1}$ whose last two coordinates are 0 and the remaining coordinates are 1. Finally by applying $v_k^{k+1}+\bar{w}_{k,k+1}^{k+1}$ and $v_{k+1}^{k+1}+\bar{w}_{k,k+1}^{k+1}$ we get the vectors u_k^{k+1} and u_{k+1}^{k+1}. Thus, after $1+(3k-6)+2=3(k+1)-6$ steps we get required vectors $u_1^{k+1}, u_2^{k+1}, \ldots, u_k^{k+1}$ and u_{k+1}^{k+1}.

Now we show that at least $3n-6$ steps are necessary. For $n \geq 3$, let $f(n)$ be the minimal possible number of steps. By induction we will show that $f(n) \geq 3n-6$.

If $n=3$ then it can be readily shown that $f(3) \geq 3$.

Suppose that for $n=k$ we have $f(k) \geq 3k-6$. Let us go over steps made for $n=k+1$. Let A be the step when the vector v_{k+1}^{k+1} was used for the first time. In this step the vector v_{k+1}^{k+1} was added to a vector $r(m)$ such that for some $1 \leq m \leq k$ mth coordinate of $r(m)$ is non-zero. In order to get the vector u_m^{k+1} whose only 0 coordinate is mth coordinate, the vector v_{k+1}^{k+1} will be used at least once more. Let B be one of these steps. Let C be step when the vector u_m^{k+1} whose only 0 coordinate is $k+1$st coordinate is obtained. In the sequence of steps made for $n=k+1$ by removing steps A, B, C and erasing $k+1$st coordinates of all vectors we can get all required vectors for $n=k$. Therefore, $f(k+1)-3 \geq f(k) \geq 3k-6$ and hence $f(k+1) \geq 3(k+1)-6$. We are done.