Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

January 2025

Problem:

Find all prime numbers p for which the number

$$3^p + 4^p + 5^p + 9^p - 98$$

has at most 6 positive divisors.

Solution: Answer: $p = 2, 3, 5$.

Let $f(p) = 3^p + 4^p + 5^p + 9^p - 98$. The primes 2,3,5 satisfy the problem conditions: $f(2) = 3 \cdot 11$, $f(3) = 7 \cdot 11^2$, $f(5) = 7 \cdot 9049$.

Let $p > 5$. Since 7 divides $3^p + 4^p$, $5^p + 9^p$ and 98 we get $7 \mid f(p)$. Now since

$$\begin{aligned}
p \equiv 1 \pmod{10} &\implies f(p) \equiv 3+4+5+9+1 \equiv 0 \pmod{11},\\
p \equiv 3 \pmod{10} &\implies f(p) \equiv 5+9+4+3+1 \equiv 0 \pmod{11},\\
p \equiv 7 \pmod{10} &\implies f(p) \equiv 9+5+3+4+1 \equiv 0 \pmod{11},\\
p \equiv 9 \pmod{10} &\implies f(p) \equiv 4+3+9+5+1 \equiv 0 \pmod{11}
\end{aligned}$$

we get that $11 \mid f(p)$. Since $f(p)$ has at most 6 positive divisors the only prime divisors of $f(p)$ are 7 and 11. On the other hand, for $p > 5$ we have $f(p) > 7 \cdot 11^2 > 7^2 \cdot 11 > 7 \cdot 11$. Therefore, there is no $p > 5$ satisfying conditions.